Variable-Player Learning for Simulation-Based Games

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Outline

- **Introduction and Motivation**
- Background
- Related Work
- Variable-Player Games
- Model: Approximating robust symmetric Nash equilibria
- Analysis: Equilibrium robustness metrics
- Experiments
- Conclusion & Ongoing Work

Introduction

- **Game theory**: branch of economics that aims to model how people or "agents" interact and make decisions

- **Machine learning**: branch of computer science which uses mathematical techniques to learn functions from data

- **Thesis**: Use machine learning to analyze large, symmetric, variable-player simulation-based games

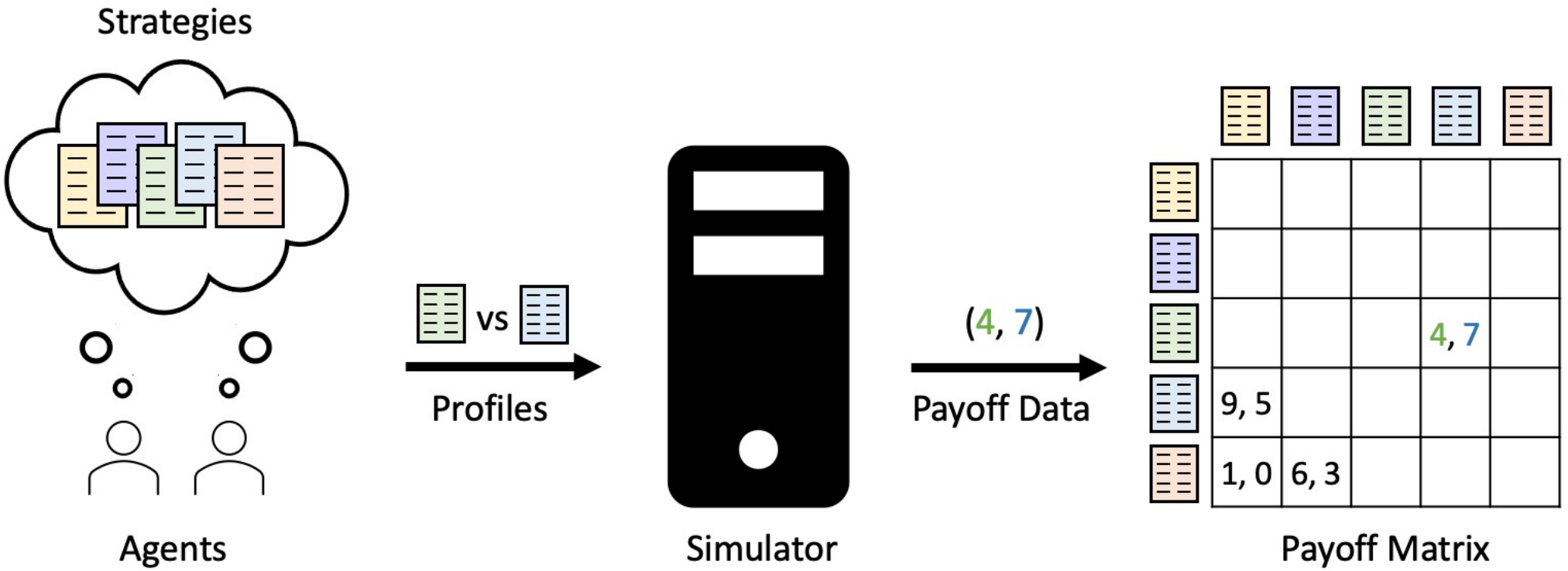
Normal-Form Games

- Type of simultaneous-move game
- Fixed set of players
- For each player
  - Set of strategies
  - Utility function
- Represented using payoff matrix

Rock-Paper-Scissors

		Player 2		
		R	**P**	**S**
Player 1	**R**	0, 0	-1, 1	1, -1
	P	1, -1	0, 0	-1, 1
	S	-1, 1	1, -1	0, 0

Simulation-Based Games



Motivation

- Applications of SBGs
  - Stock market
  - Cybersecurity
  - Credit networks
  - Trading agent competitions
- Likely that the number of players is **large** and **unknown**

https://www.losaltosonline.com/news/sections/business/185-business-columns/62148-how-the-stock-market-has-reacted-to-pandemics

Sokota et al. [11]

Thesis

Thesis Essential Questions

- How do we construct a variable-player game-theoretic model?

- How do we analyze a variable-player game-theoretic model?

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  - **Deep learning**
  - **Game theory**
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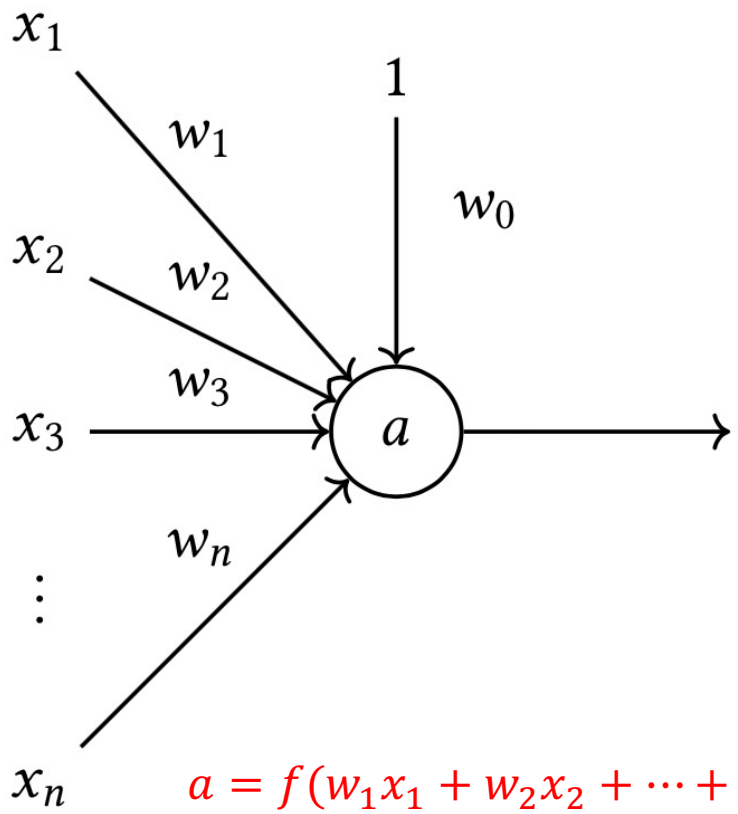
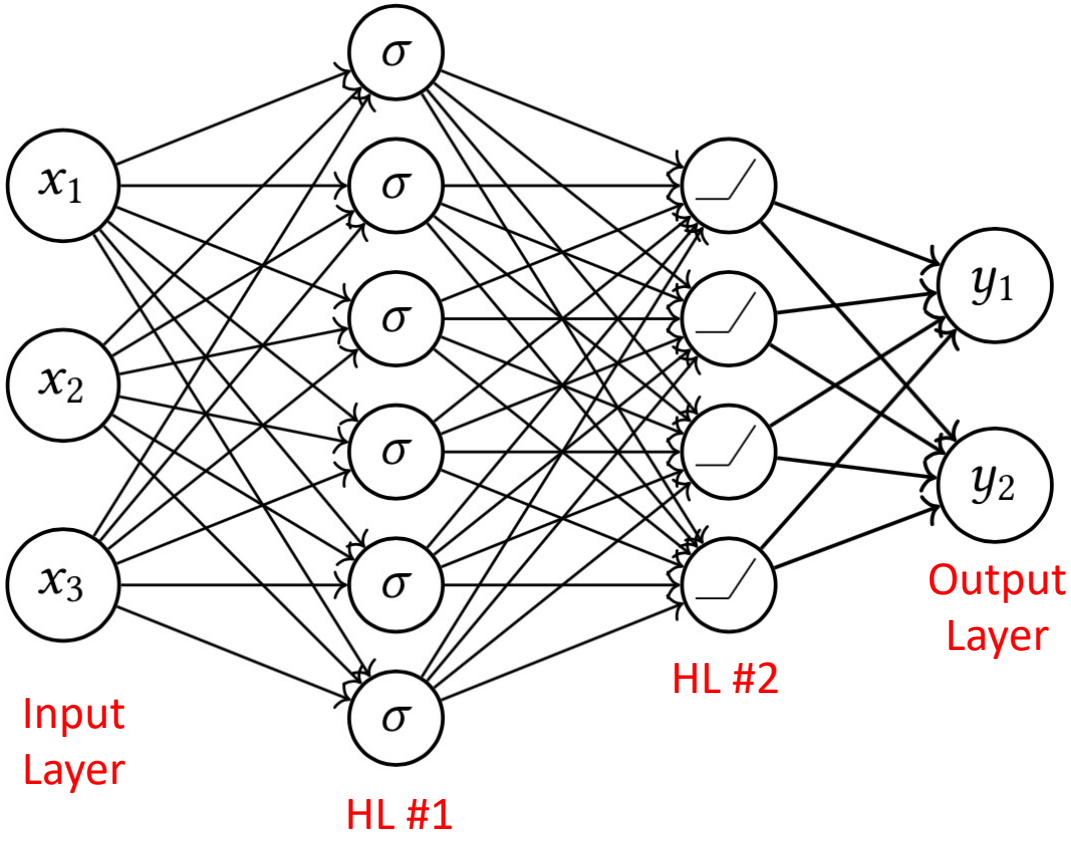
Background: Deep Learning

Artificial Neuron



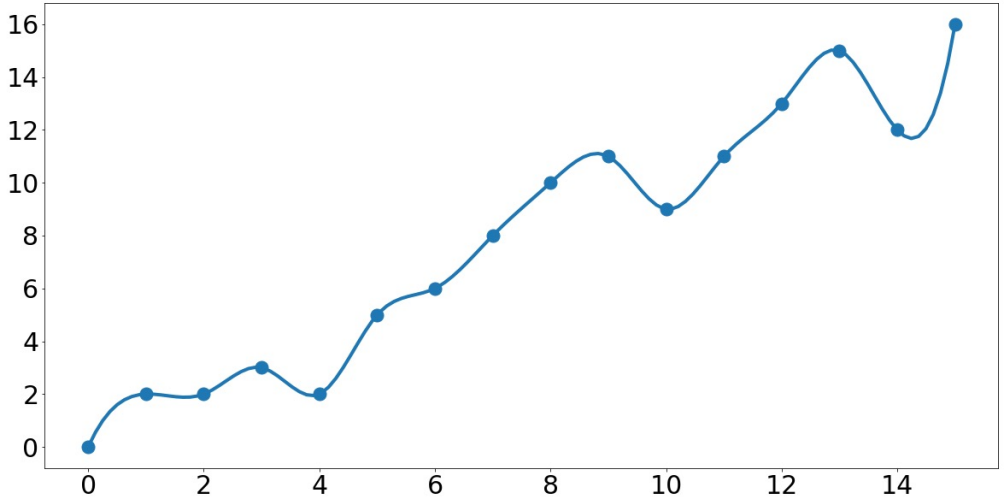
$$a = f(w_1x_1 + w_2x_2 + \cdots + w_nx_n)$$

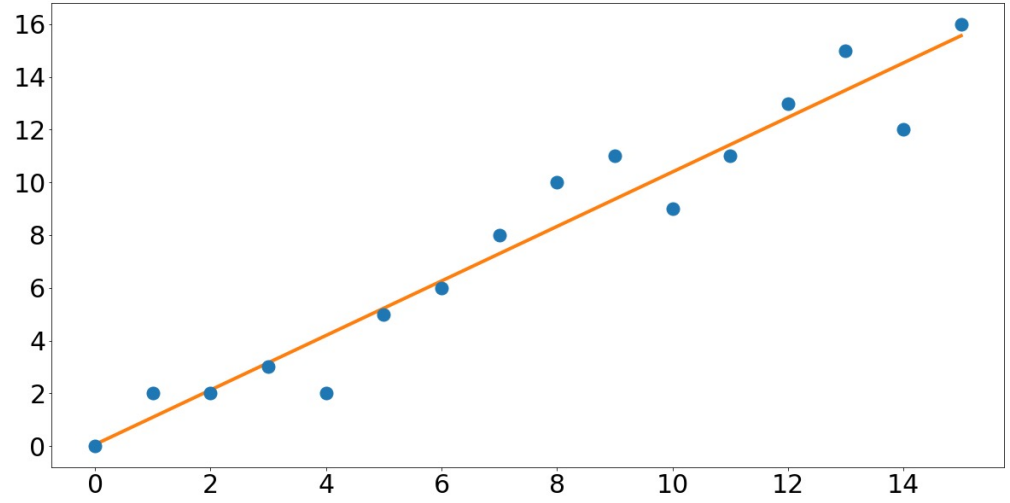
Neural Network



Background: Deep Learning (*cont.*)

- **Training phase:** Optimize neural network *parameters*
  - Objective: minimize error
- **Validation phase:** Optimize neural network *hyperparameters*
  - Objective: model generalizes well to new data
- **Testing phase:** Make predictions on new, unlabeled data





Background: Game Theory

- A **normal-form game** is a tuple $\tau_n = (P, S, u)$ where
  - $P = \{1, \ldots, n\}$
  - $S = S_1 \times \cdots \times S_n$
  - $u : S \mapsto \mathbb{R}^n$

Rock-Paper-Scissors

- $P = \{1, 2\}$
- $S_1 = \{\mathbf{R}, \mathbf{P}, \mathbf{S}\};\ S_2 = \{\mathbf{R}, \mathbf{P}, \mathbf{S}\}$
- $S = S_1 \times S_2$
- $S = \{(\mathbf{R}, \mathbf{R}), (\mathbf{R}, \mathbf{P}), (\mathbf{R}, \mathbf{S}),$
  $(\mathbf{P}, \mathbf{R}), (\mathbf{P}, \mathbf{P}), (\mathbf{P}, \mathbf{S}),$
  $(\mathbf{S}, \mathbf{R}), (\mathbf{S}, \mathbf{P}), (\mathbf{S}, \mathbf{S})\}$
- u is represented by payoff matrix

Types of Strategies

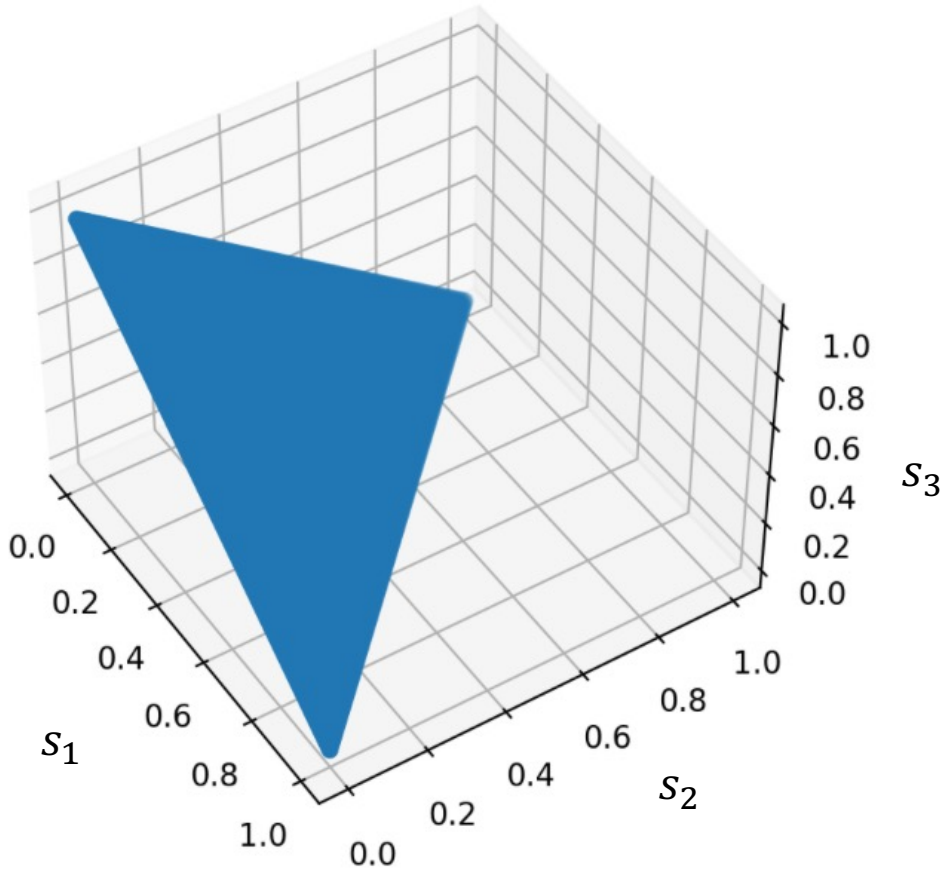
- **Pure strategy:** any $s \in S_i$
- **Pure-strategy profile:** $\vec{s} = (s_1, \dots, s_n)$
- **Mixed strategy:** probability distribution over actions, denoted by σ
- **Mixed-strategy profile:** $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$

Rock-Paper-Scissors

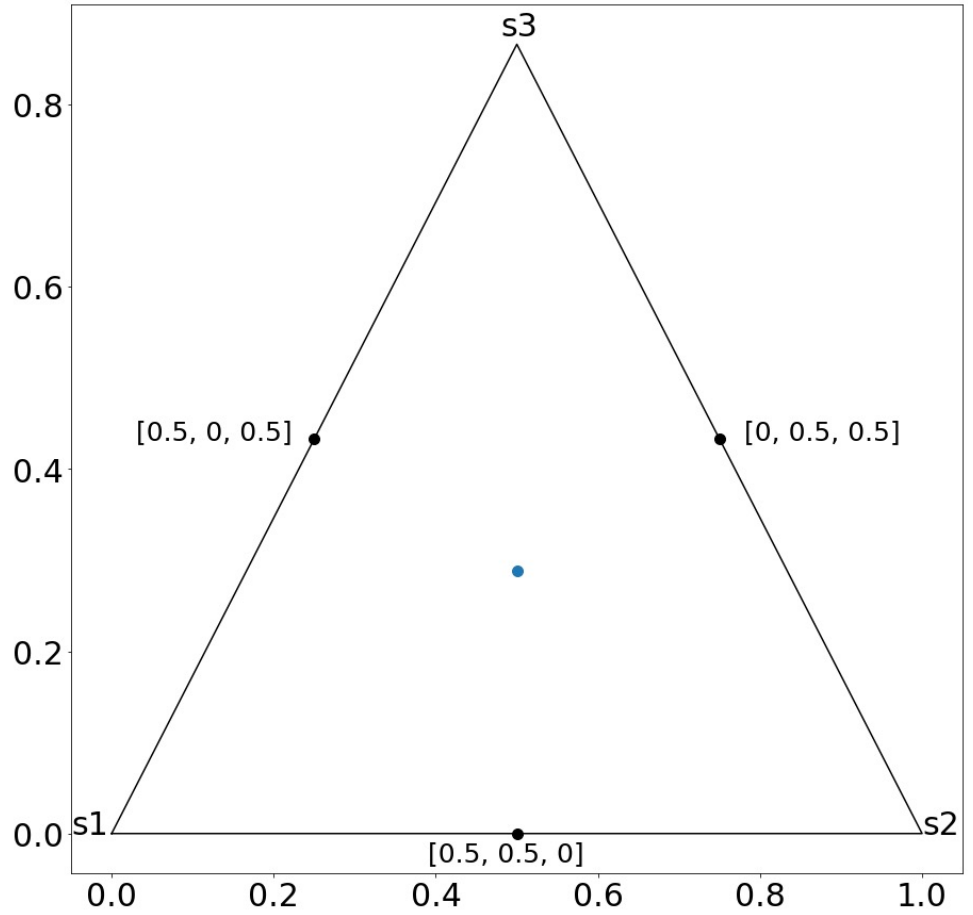
		Player 2		
		1/3 R	1/3 P	1/3 S
Player 1	1/3 R	0, 0	-1, 1	1, -1
	1/3 P	1, -1	0, 0	-1, 1
	1/3 S	-1, 1	1, -1	0, 0

$\vec{\sigma}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\vec{\sigma}_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Set of all RPS mixed strategies described by 2-simplex



3D



2D Projection

Expected Utility

- **Expected utility:**

$$u_i(\vec{\sigma}) = \sum_{s \in S} u_i(s) \prod_{j=1}^{n} \vec{\sigma}_j(s_j)$$

$$\begin{aligned} u_1(\vec{\sigma}_2) = &\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \cdot 0 + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \cdot -1 + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \cdot 1 \\ &+ \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) \cdot 1 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \cdot 0 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) \cdot -1 \\ &+ \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \cdot -1 + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \cdot 1 + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \cdot 0 \\ &= 0 \end{aligned}$$

Rock-Paper-Scissors

Player 1 \ Player 2	1/4 R	1/2 P	1/4 S
1/4 R	0, 0	-1, 1	1, -1
1/2 P	1, -1	0, 0	-1, 1
1/4 S	-1, 1	1, -1	0, 0

$$\vec{\sigma}_2 = \left(\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)\right)$$

Deviation Payoffs

- Define $\vec{\sigma}_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$
- **Deviation payoff:**

$$\text{devPay}_i(s, \vec{\sigma}) = u_i(s, \vec{\sigma}_{-i})$$

- **Deviation payoff function,** $\text{devPay}(\vec{\sigma})$:

$$(\text{devPay}_1(s_1, \vec{\sigma}), \dots, \text{devPay}_n(s_{|s_n|}, \vec{\sigma}))$$

$$\text{devPay}_1(\mathbf{R}, \vec{\sigma}_2) = \left(\frac{1}{4}\right) \cdot 0 + \left(\frac{1}{2}\right) \cdot -1 + \left(\frac{1}{4}\right) \cdot 1 = \mathbf{-0.25}$$

$$\text{devPay}_1(\mathbf{P}, \vec{\sigma}_2) = \mathbf{0}$$

$$\text{devPay}_1(\mathbf{S}, \vec{\sigma}_2) = \mathbf{0.25}$$

Rock-Paper-Scissors

Player 1 \ Player 2	1/4 R	1/2 P	1/4 S
1/4 R	0, 0	-1, 1	1, -1
1/2 P	1, -1	0, 0	-1, 1
1/4 S	-1, 1	1, -1	0, 0

$$\vec{\sigma}_2 = \left(\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)\right)$$

Nash Equilibrium

- **Nash equilibrium:** a set of strategies such that no player can gain by deviating
- **Nash's Theorem:** *Every finite game with two or more players contains at least one Nash equilibrium.*
- In RPS, $\vec{\sigma}_1 = (\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right))$ is a Nash equilibrium

Rock-Paper-Scissors

		Player 2		
		1/3 R	1/3 P	1/3 S
Player 1	1/3 R	0, 0	-1, 1	1, -1
	1/3 P	1, -1	0, 0	-1, 1
	1/3 S	-1, 1	1, -1	0, 0

$$\vec{\sigma}_1 = (\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right))$$

Regret

- **Player regret:**

$$\varepsilon_i(\vec{\sigma}) = \underbrace{\max_{s \in S} u_i(s, \vec{\sigma}_{-i})}_{\text{max deviation payoff}} - \underbrace{u_i(\vec{\sigma})}_{\text{exp util}}$$

- **Regret:**

$$\varepsilon(\vec{\sigma}) = \max_{i \in P} \varepsilon_i(\vec{\sigma})$$

- $\varepsilon(\vec{\sigma}_1) = 0, \varepsilon(\vec{\sigma}_2) = 0.25$
- **ε-Nash equilibrium:** $\vec{\sigma}$ such that $\varepsilon(\vec{\sigma}) \leq \varepsilon$ for some ε

Rock-Paper-Scissors

Player 1 \ Player 2	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

$$\vec{\sigma}_1 = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$$

$$\vec{\sigma}_2 = \left(\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)\right)$$

Symmetric Games and Equilibria

- A **symmetric game** is a tuple $\tilde{\tau}_n = (n, S, u)$ where
  - n is the number of players
  - $S = \{s_1, \dots, s_k\}$
  - $C = \{\vec{c} \in \mathbb{Z}^k : \vec{c}_j \geq 0, \sum_{j=1}^{k} \vec{c}_j = n\}$
  - $u : C \mapsto \mathbb{R}^k$
- **Symmetric Nash equilibrium:** Nash equilibrium in which all players are playing the same (pure- or mixed-) strategy

Rock-Paper-Scissors

Player 2 \ Player 1	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Motivation: Computing Approximate NE

- Computational complexity of computing exact NE
- Representational challenges
- Approximate Nash-finding algorithms
  - Lemke-Howson
  - Govindan-Wilson (global Newton method)
  - Fictitious play
  - Replicator dynamics

Replicator Dynamics

REPLICATORDYNAMICS(Γ_n, $\vec{\sigma}$, ϵ, numIters):

offset $\leftarrow$ [MIN(Γ_n, i) **for** each player i]

repeat

$\vec{\sigma} \leftarrow \vec{\sigma} \cdot$ (DEVPAYS(Γ_n, $\vec{\sigma}$) - offset)

$\vec{\sigma} \leftarrow \vec{\sigma} \,/\, \sum(\vec{\sigma})$

until numIters

if REGRET(Γ, $\vec{\sigma}$) $\leq \epsilon$

return $\vec{\sigma}$

return -1

Learned Deviation Payoff Function

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Related Work

- Learning game models from data ([1], [7], [10], [14], [18])
  - Learning Deviation Payoffs in Simulation-Based Games [11]*
- Games with a variable number of players
  - Simulation-based game theory ([2], [9], [15], [16], [17], [19])
  - Other examples ([3], [4], [6], [8], [12], [13])

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Variable-Player Symmetric Games

- A **variable-player symmetric game** is a tuple $\tilde{\tau}_{mn} = (P, S, u)$ where
  - $P = \{m, \dots, n\}$
  - $S = \{s_1, \dots, s_k\}$
  - $C_{mn} = \{\vec{c} \in \mathbb{Z}^k : \vec{c}_j \geq 0, m \leq \sum_{j=1}^{k} \vec{c}_j \leq n\}$
  - $u : C_{mn} \mapsto \mathbb{R}^k$

Variable-Player Symmetric Games (*cont.*)

- An **instance** of a variable-player symmetric game is a tuple $\tilde{\tau}_p = (p, S, u)$ where
  - p is the number of players, with $m \leq p \leq n$
  - $S = \{s_1, \ldots, s_k\}$
  - $C_p = \{\vec{c} \in \mathbb{Z}^k : \vec{c}_j \geq 0,\ \sum_{j=1}^{k} \vec{c}_j = p\}$
  - $u : C_p \mapsto \mathbb{R}^k$

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- Introduction and Motivation
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- **Model**
  - **Approximating deviation payoffs**
  - **Approximating symmetric Nash equilibria**
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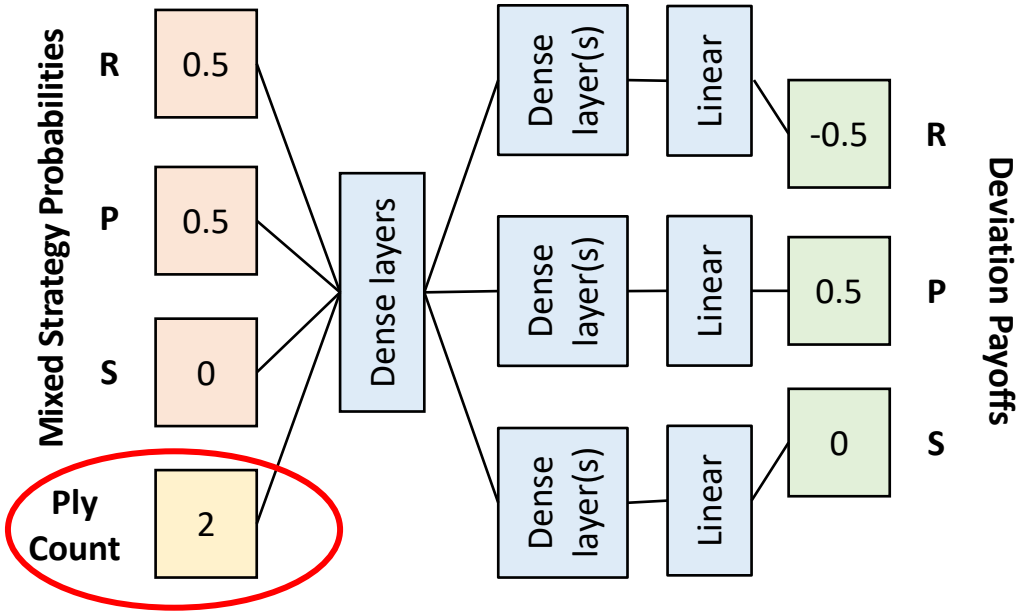
Model: Approximating deviation payoffs

- **Hypothesis:** the payoffs in a game with x players are similar or related to the payoffs in the same game with $x \pm 1$ players, given a large value of x
- We use a **multi-headed neural network** to learn a mapping from mixed-strategy profiles and number of players to deviation payoffs

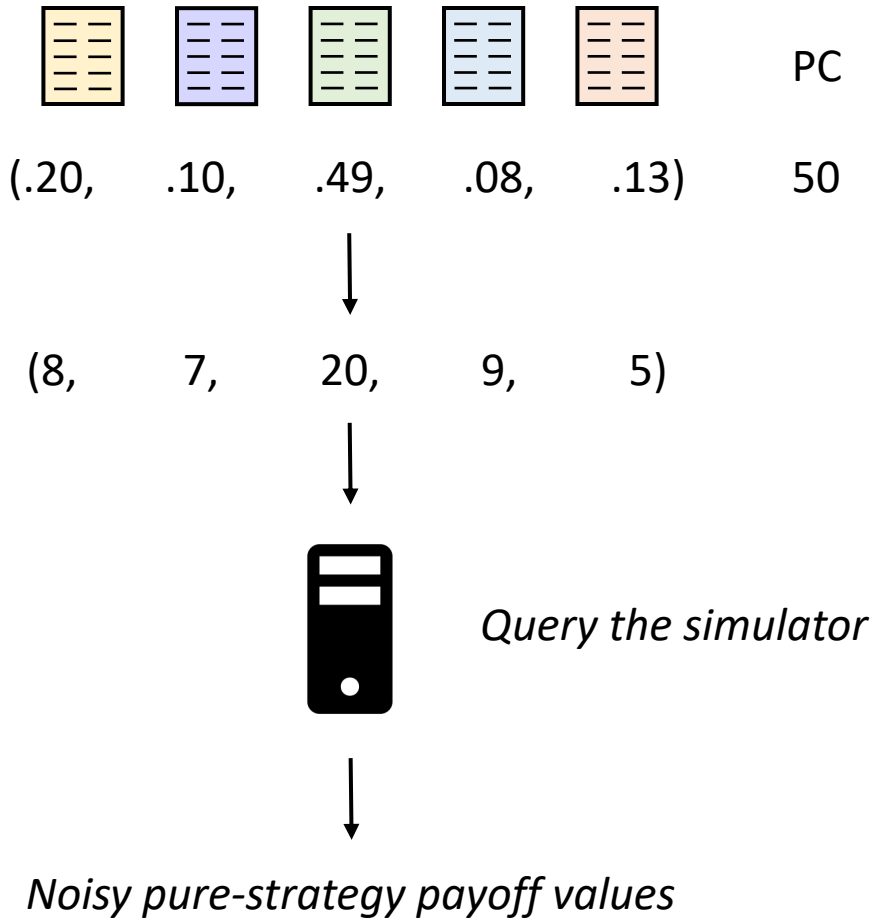


Neural Network Architecture for Variable-Player Learning

Model: Approximating deviation payoffs (*cont.*)

Generating training data:

- Generate random mixed-strategy profile
  - Dirichlet distribution
- Generate random player count
  - Representative instances
  - Uniform random across entire range
- Sample a pure-strategy profile according to mixed-strategy profile for each opponent
- Query simulator for PS payoffs



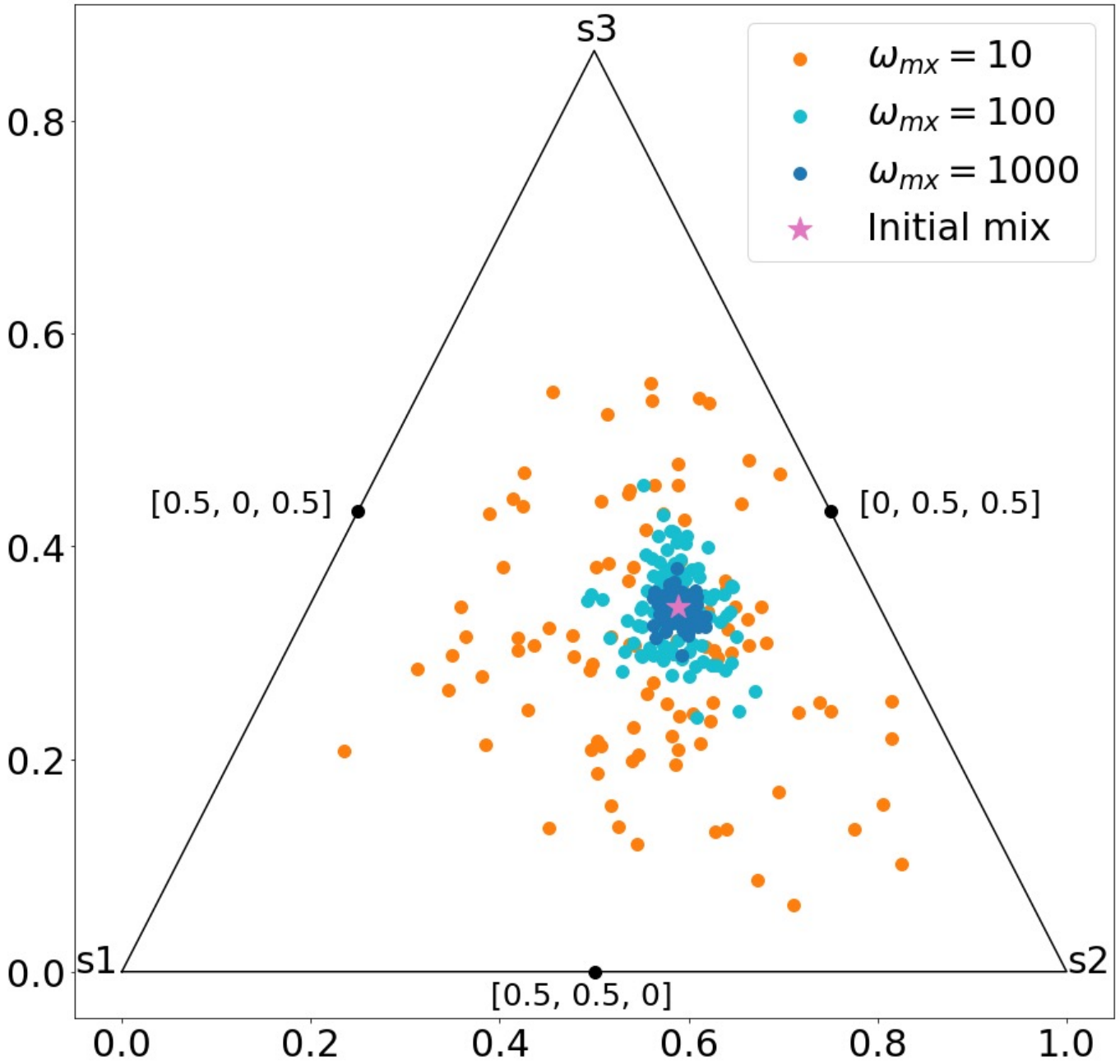
Model: Approximating symmetric NE

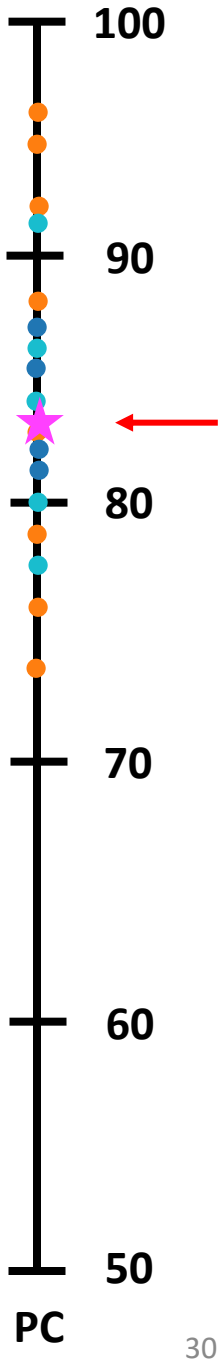
- Want to focus learning on areas of simplex where we think there might be approximate Nash equilibria

Algorithm overview:

- For *i* iterations
  - Run Nash-finding algorithm
  - Sample in neighborhood of candidate Nash and corresponding player counts
  - Retrain network, adding resamples to training data

Neighborhood Sampling





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  - **Equilibrium robustness metrics**
  - **Comparison of robustness metrics**
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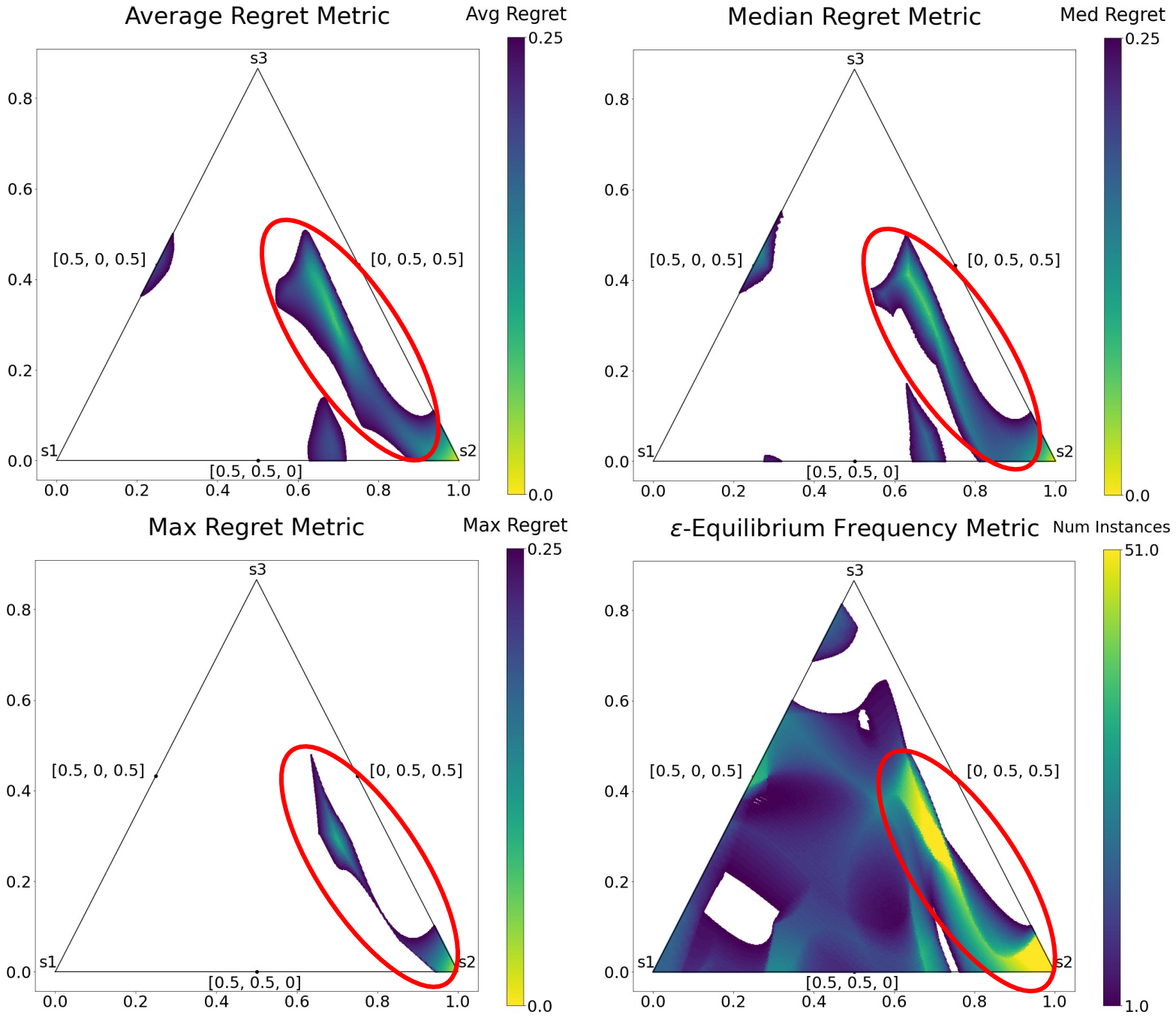
Analysis: Equilibrium robustness metrics

- Typical game-theoretic analysis: find approximate NE in games with fixed number of players
- Finding approximate NE in game with variable number of players is not as straightforward
- Robustness: measure of how well an equilibrium generalizes across all instances of game
- Several proposed robustness metrics
  - Average regret
  - Median regret
  - Max regret
  - Approximate equilibrium frequency

Analysis: Comparison of robustness metrics



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  - **Random game generation**
  - **Comparison to existing work**
  - **Approximating symmetric Nash equilibria**
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Experiments

- Does our technique outperform Sokota et al.'s technique on variable-player games?

- Is the iterative refinement necessary for the variable-player learning model?

Experiments: Random game generation

- Random additive polynomial sine BAGG-FNAs serve as a proxy for simulator data
- Payoff functions are complex but learnable
- Compact representation -> simplifies ground truth deviation payoff calculation
- Can be easily defined with a variable number of players

Experiment 1: Experimental specification

Random Games

- 250 random symmetric games
- Range: 50 to 100 players
- 5 strategies

Evaluation

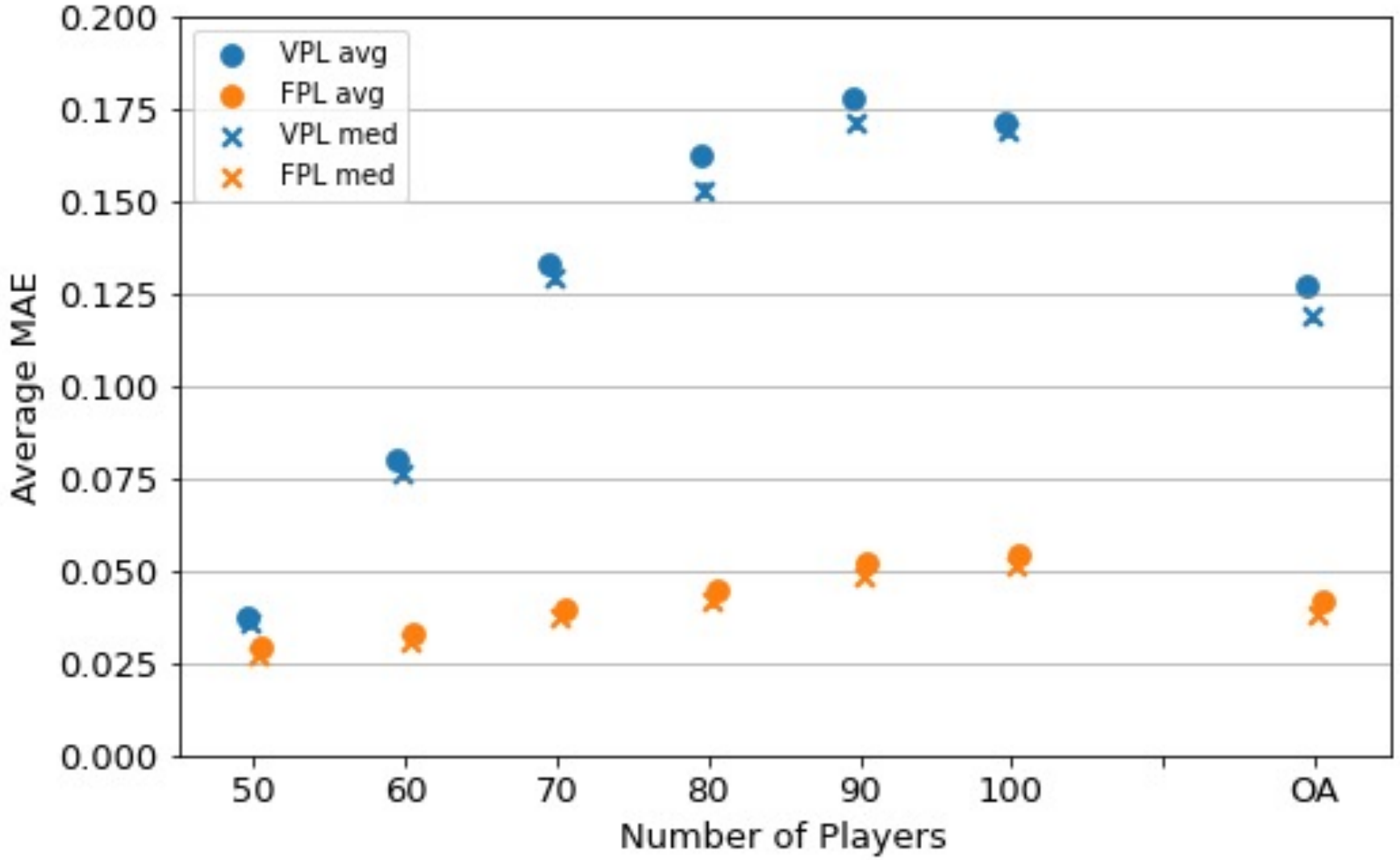
- Average deviation payoff MAE across all strategies

Learning Models

- Fixed-Player Learning (FPL)
  - Train 6 NNs with player counts: 50, 60, 70, 80, 90, 100
- Variable-Player Learning (VPL)
  - Train 1 NN which learns across range of player counts
- Hyperparameters optimized separately

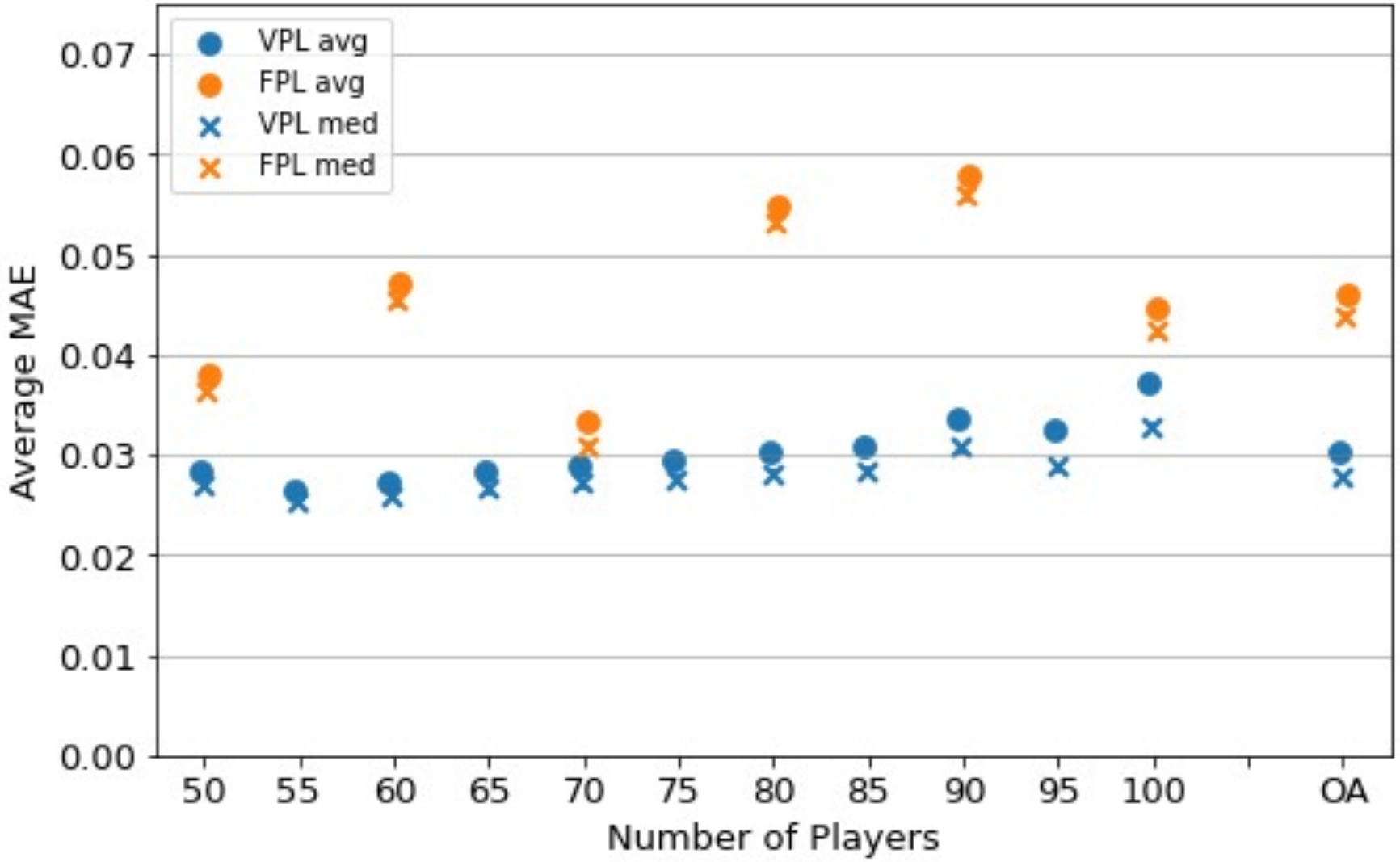
Experiment 1: Comparison to Existing work

PC: Representative instances



Experiment 1: Comparison to Existing work

PC: Uniform across entire range



Experiment 2: Experimental specification

Random Games

- 500 random symmetric games
- Range: 50 to 100 players
- 5 strategies

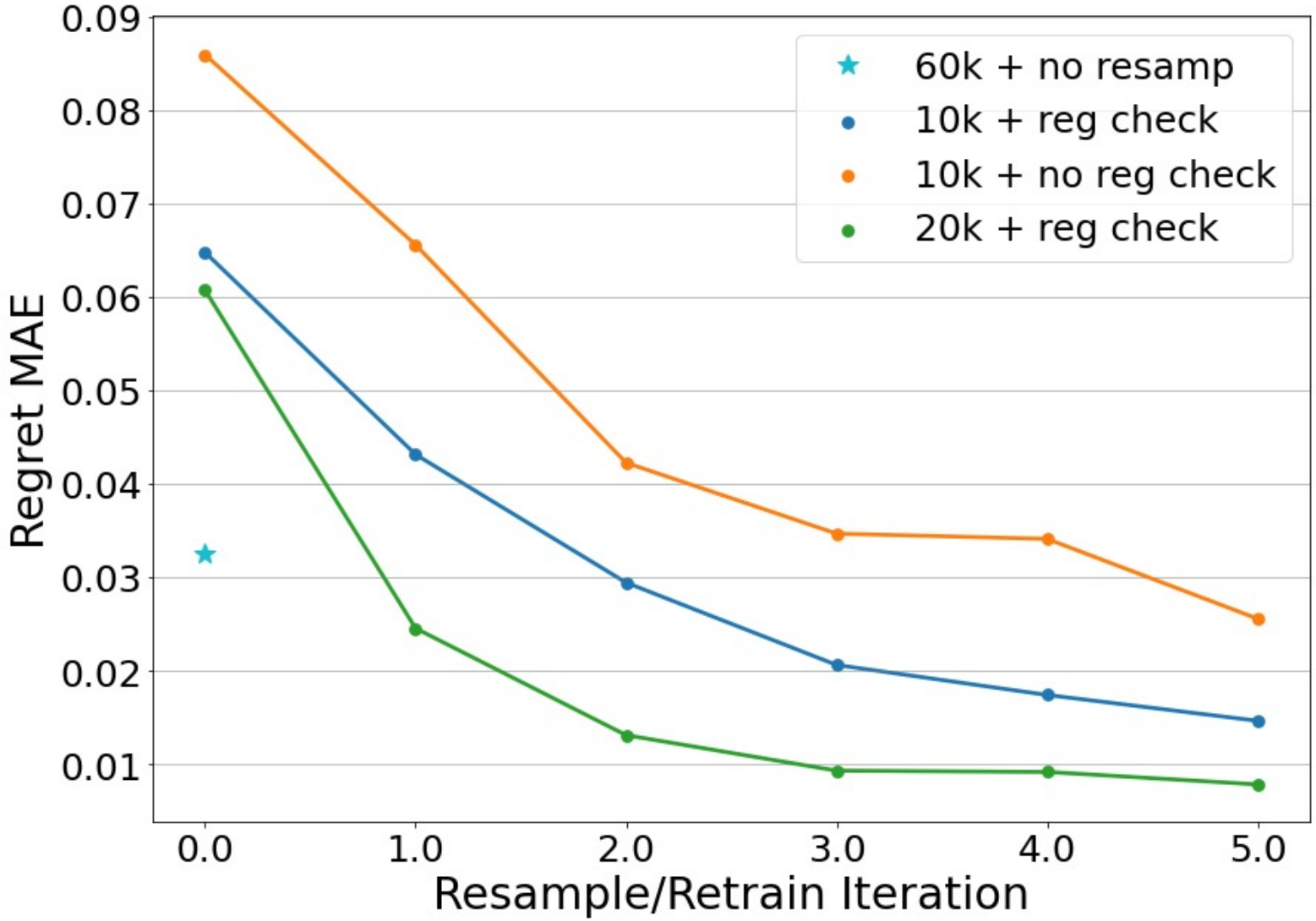
Evaluation

- Regret MAE

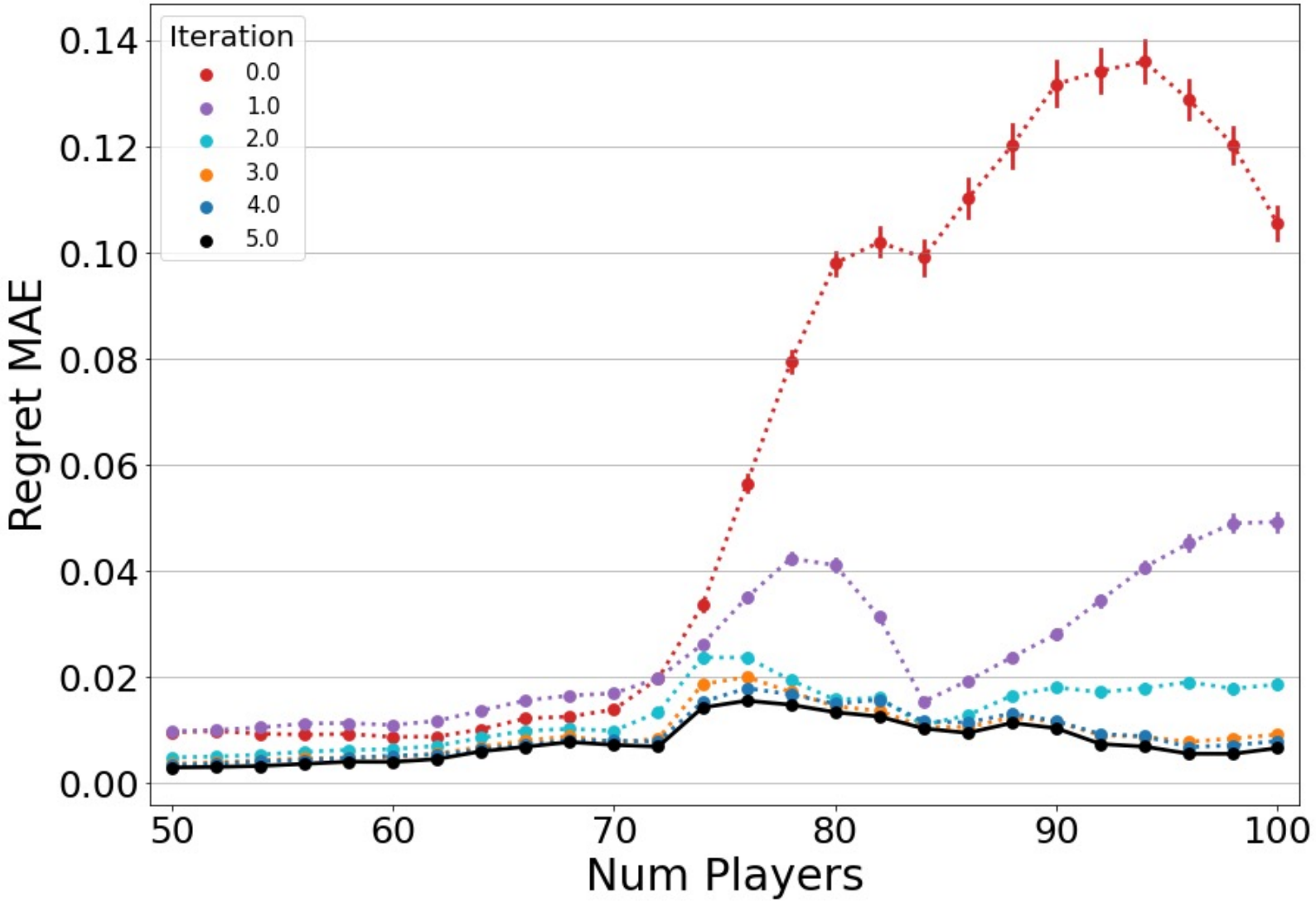
Learning Models

- Model 1
  - No resampling
- Model 2 (baseline)
  - Resampling and intermediate regret check
- Model 3
  - Resampling and no intermediate regret check
- Model 4
  - Resampling and intermediate regret check

Experiment 2: Approximating Symmetric NE



Experiment 2: Approximating Symmetric NE



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Conclusion

- Motivation: many real-world interactions of interest have a large, uncertain number of players
- Current game-theoretic analysis: limited to fixed-player games
- Related work: number of players is hyperparameter and/or small range analyzed
- **Main contribution: new type of analysis which accommodates uncertainty in the number of players**

Ongoing Work

- Scalability experiments
  - Size of range
  - Magnitude of player counts
  - Number of strategies
- Evaluate performance on a wider range of games
- Extend this technique to analyze variable-player role-symmetric games
- Extend this technique to vary parameters of simulation environment
- Theoretical guarantees?

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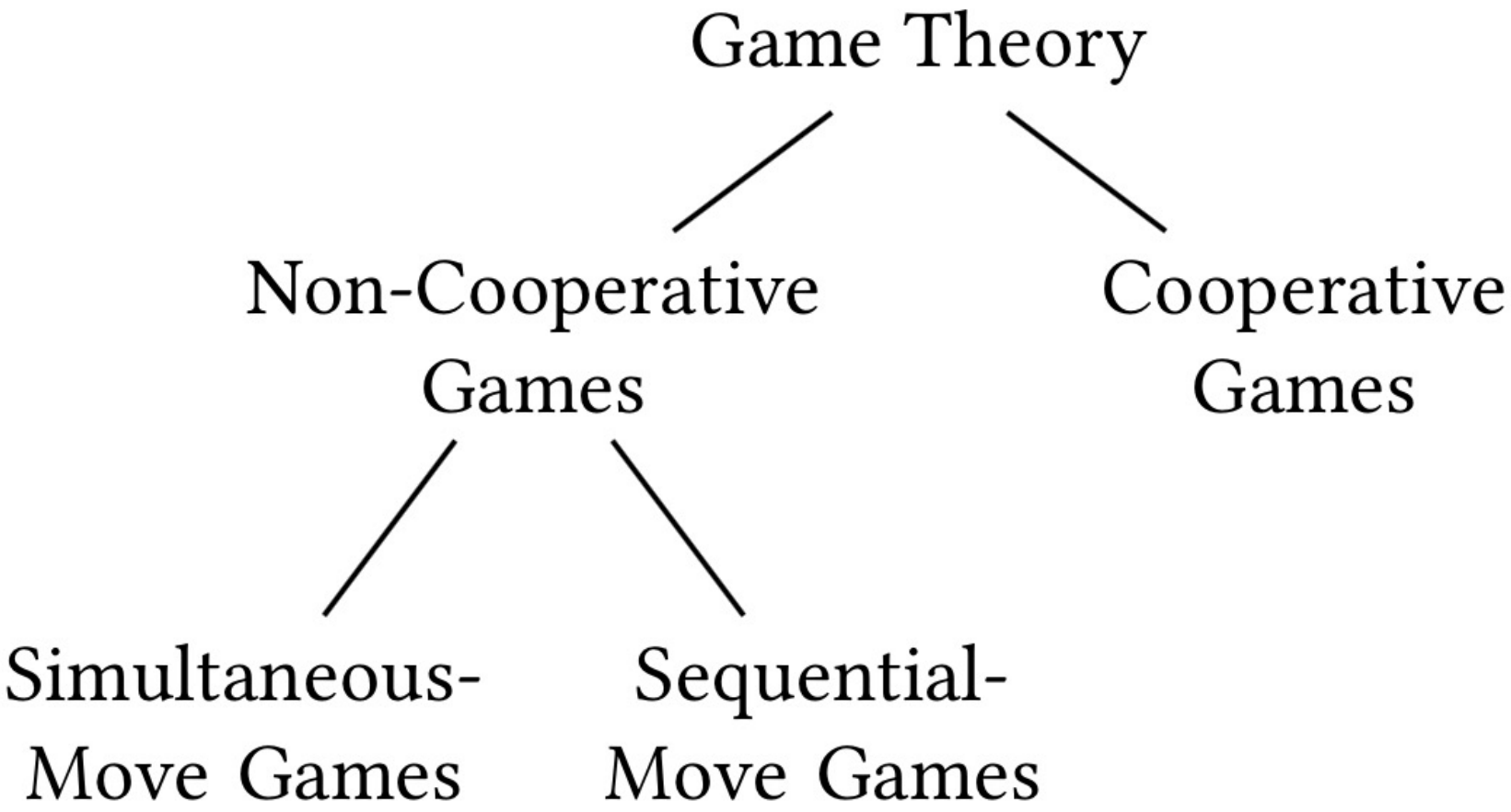
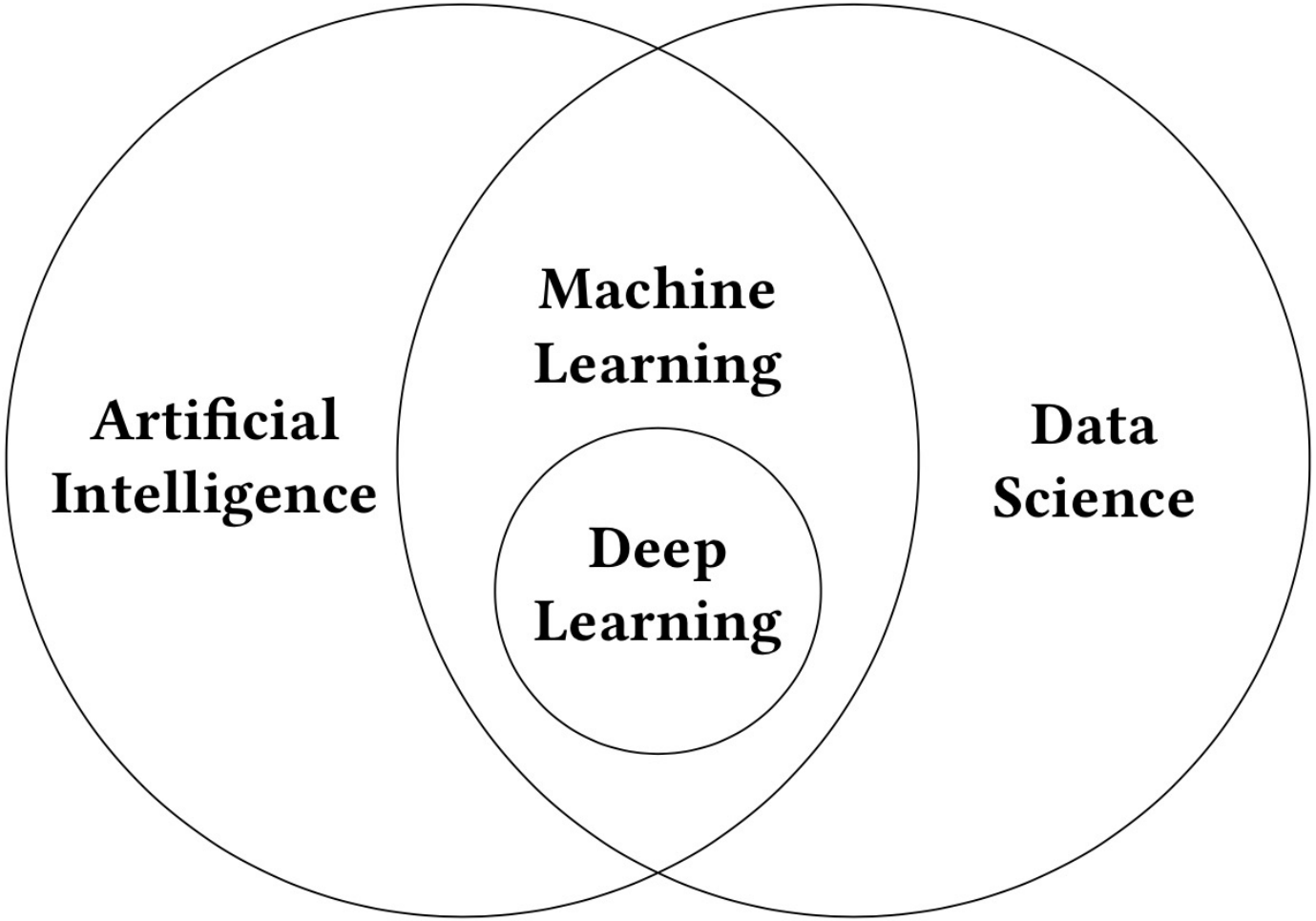
Thank you!

Hierarchy of Game Classes



Machine Learning Venn Diagram



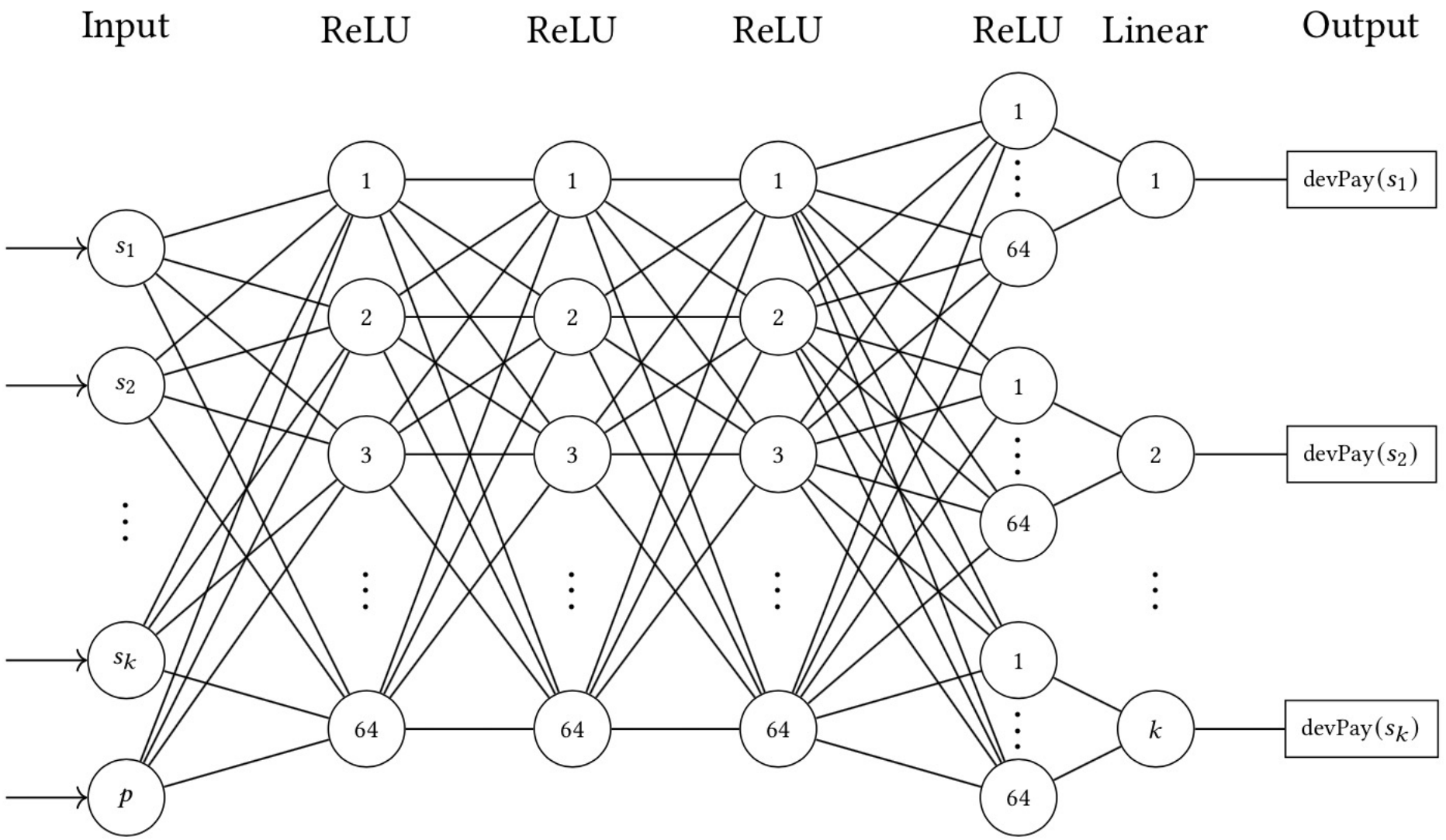
Common Activation Functions

Function Name	Function Definition
Linear	$f(x) = x$
Sigmoid (σ)	$f(x) = \frac{1}{1+e^{-x}}$
Rectified Linear Unit (ReLU)	$f(x) = \max(0, x)$
Hyperbolic Tangent (tanh)	$f(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

Common Hyperparameters

- Number of hidden layers
- Number of nodes per layer
- Type(s) of layers
- Choice of activation function
- Optimizer
- Learning rate
- Batch size
- Number of epochs
- Weight initialization
- Learning rate decay
- Batch normalization
- Regularization

Variable-Player Learning (VPL) Architecture

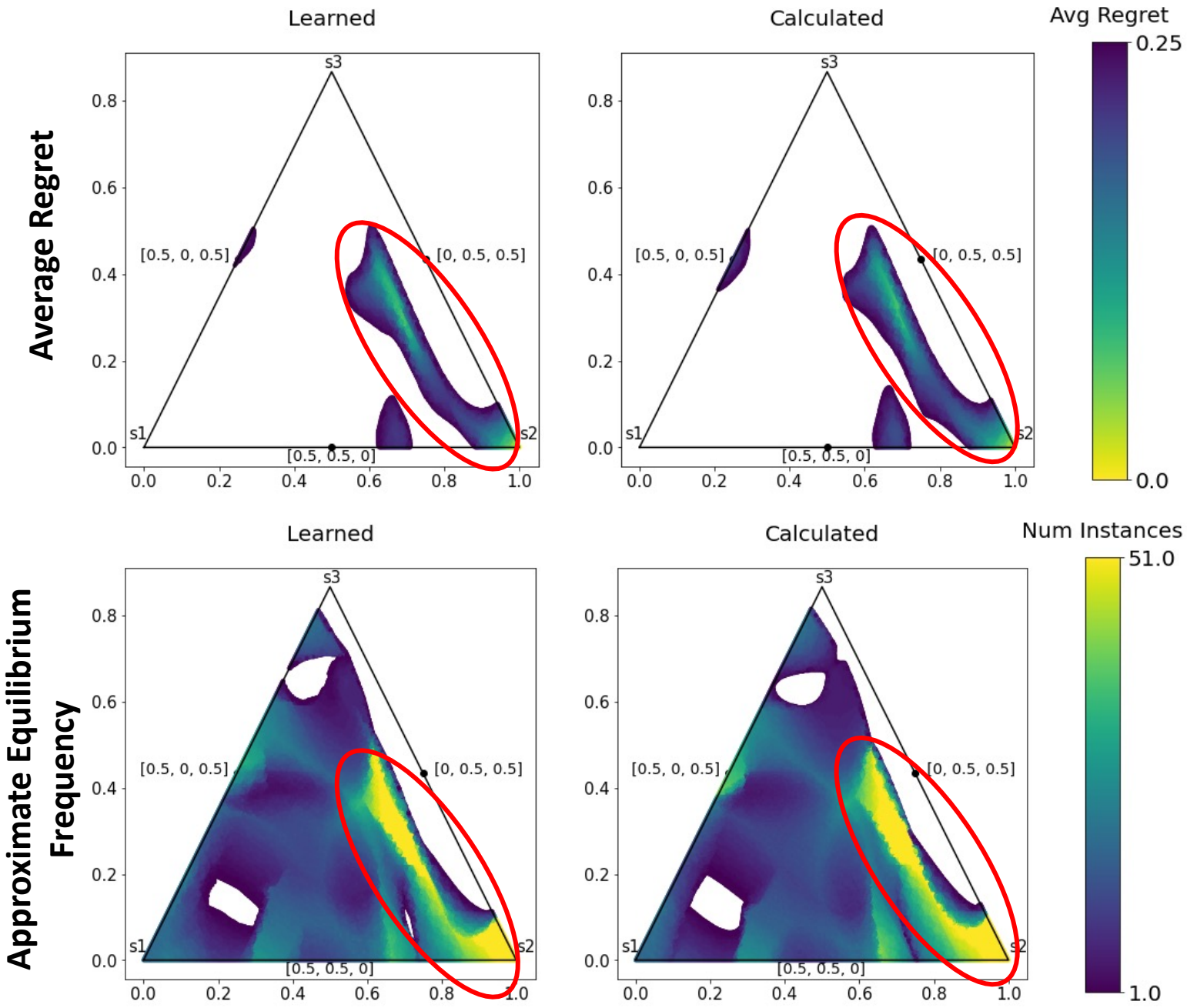


Model: Approximating robust symmetric NE

AppxRobustNashEquilibria(numInitQueries, numResampQueries, numIters, $\vec{\alpha}$, $\vec{\omega}$, m, n):

$[\vec{\sigma}] \leftarrow \text{Dir}(\vec{\alpha}, \text{numInitQueries})$
$[p] \leftarrow \text{Uniform}([\vec{\sigma}], m, n)$
$[\vec{s}] \leftarrow \text{sampleOppProfiles}([\vec{\sigma}], [p])$
$[\vec{u}] \leftarrow \text{samplePSPayoffs}([\vec{s}])$
$\text{data} \leftarrow ([\vec{\sigma}], [p], [\vec{u}])$
regressor.fit(data)
repeat
    $([\vec{\sigma}^*], [p^*]) \leftarrow \text{findNash}(\text{regressor})$
    $([\vec{\sigma}], [p]) \leftarrow \text{sampleNbhd}([\vec{\sigma}^*], [p^*], \vec{\omega}, \text{numResampQueries})$
    $[\vec{s}] \leftarrow \text{sampleOppProfiles}([\vec{\sigma}], [p])$
    $[\vec{u}] \leftarrow \text{samplePSPayoffs}([\vec{s}])$
    $\text{data} \leftarrow \text{data} + ([\vec{\sigma}], [\vec{s}], [\vec{u}])$
    regressor.fit(data)
until numIters
$([\vec{\sigma}^*], [p^*]) \leftarrow \text{findNash}(\text{regressor})$
return $\text{findRobustNash}([\vec{\sigma}^*], [p^*])$

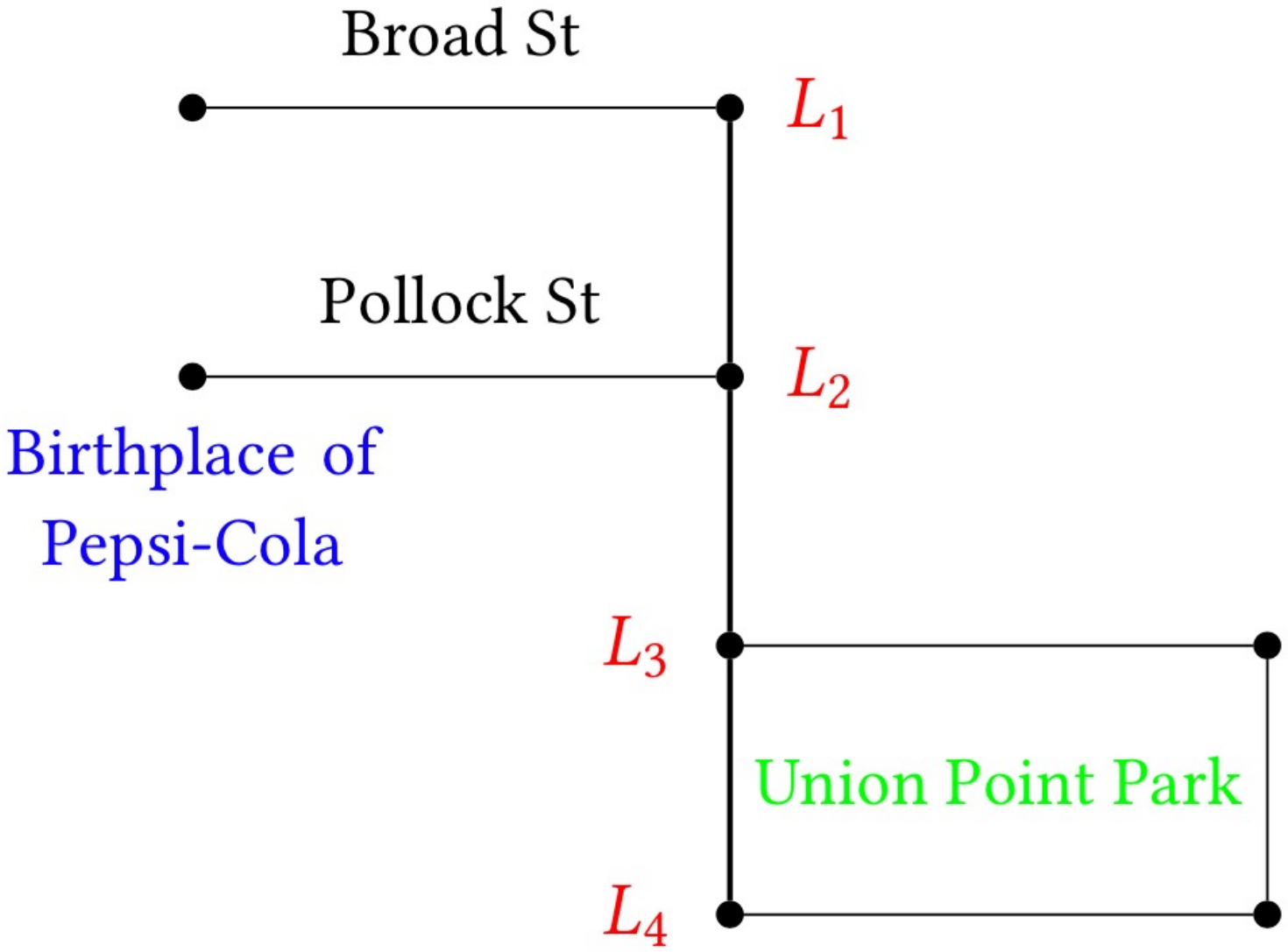
Analysis: Comparison of robustness metrics



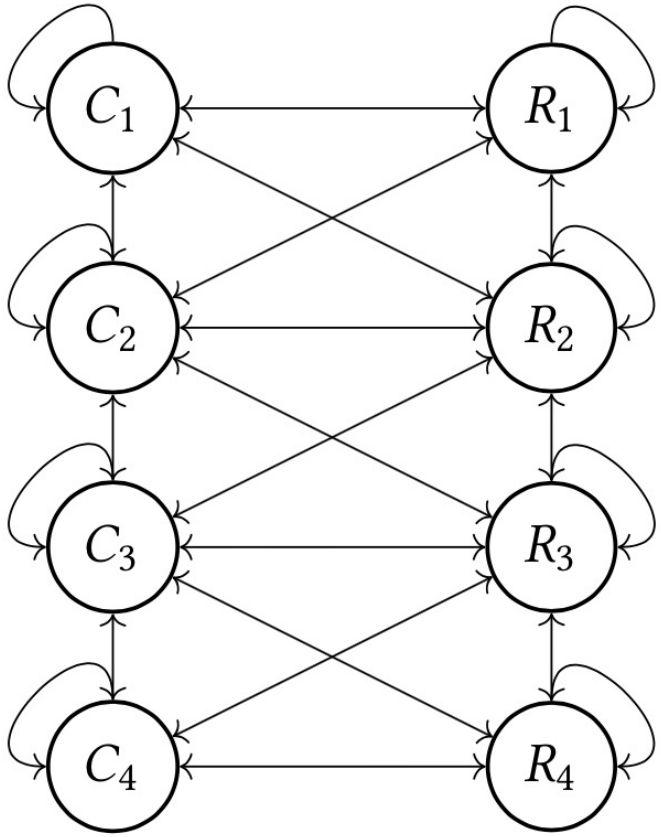
Experiments: Random game generation

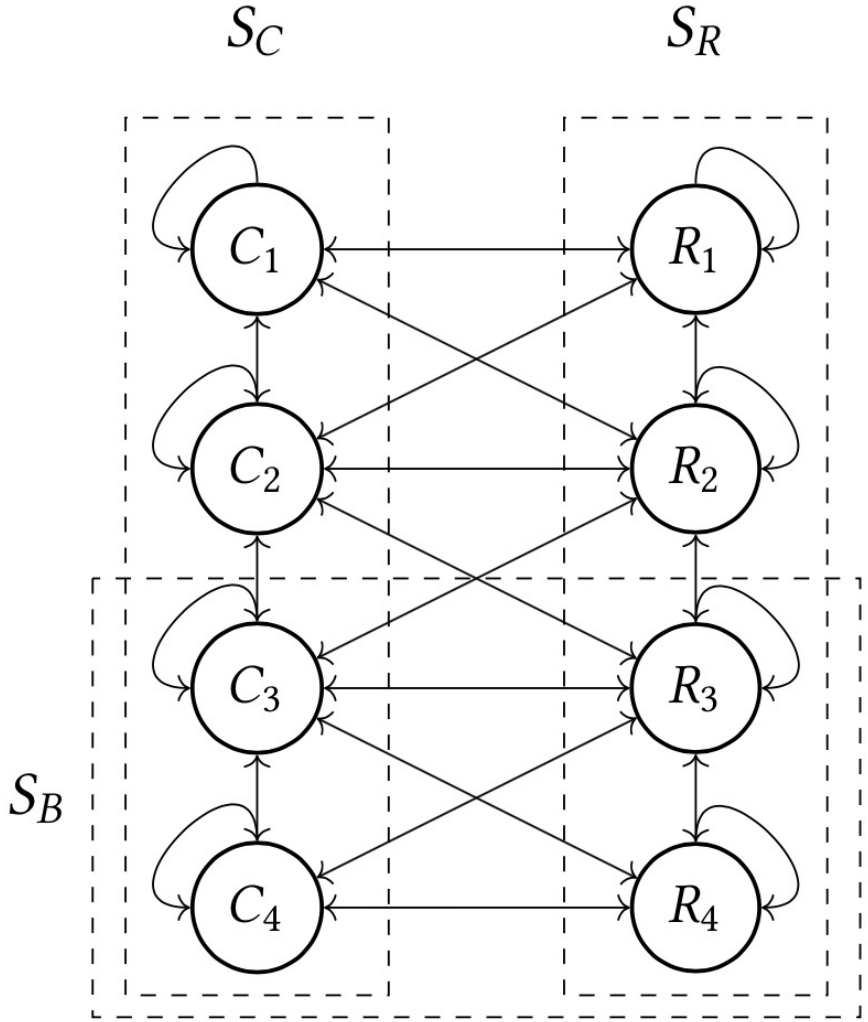
- Action-graph games ([5])
  - Actions are represented as vertices
  - Edges encode dependence among actions
  - Players have access to a subset of the actions
- AGGs can be directed or undirected
- To compute payoffs—only necessary to look at configuration of neighborhood
- Most useful when game exhibits player symmetries

AGG Example


Broad St
L_1
Pollock St
L_2
Birthplace of
Pepsi-Cola
L_3
Union Point Park
L_4

AGG Example (cont.)



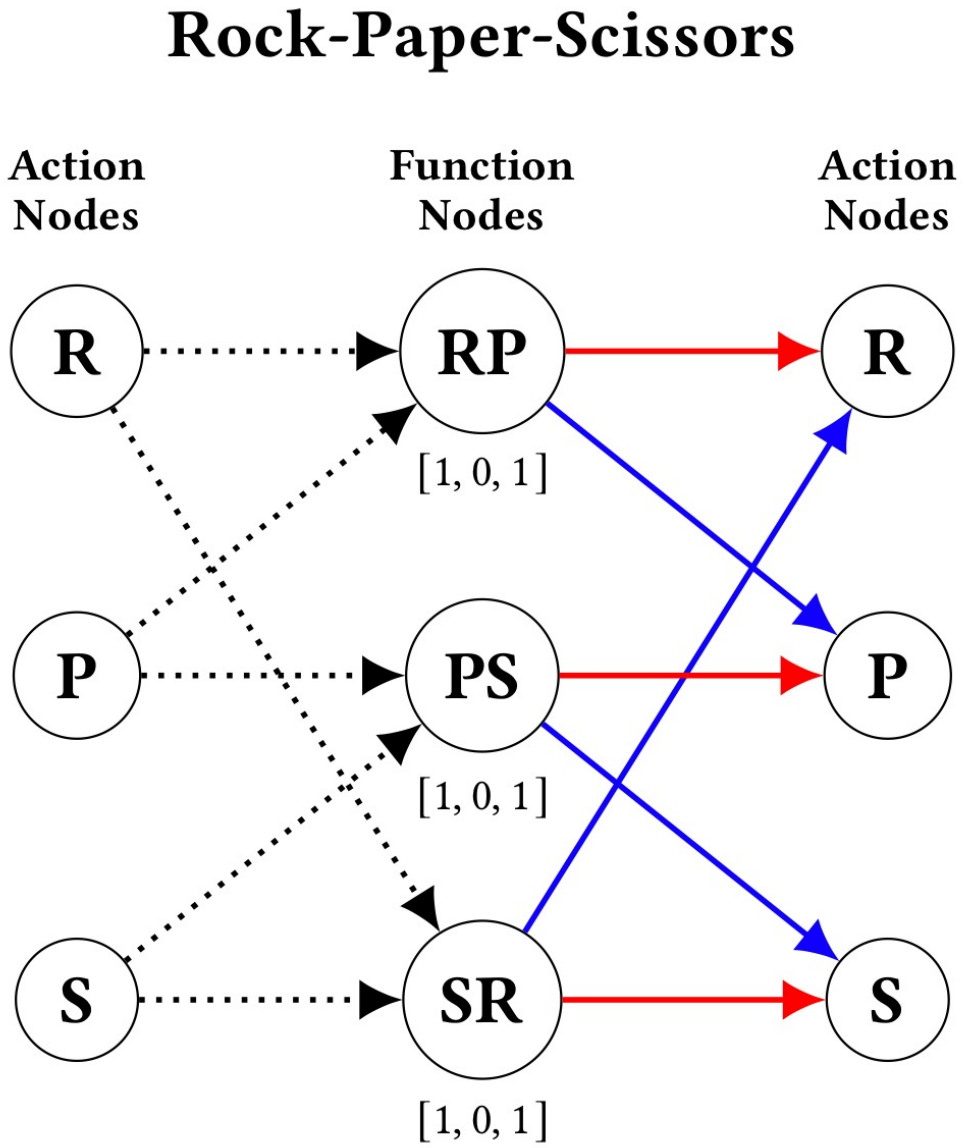


Experiments: Random game generation

- BAGG-FNAs
  - Nodes of graph can be partitioned into two independent sets (action nodes and function nodes)
  - Function nodes
    - Input: total number of players playing actions in neighborhood of given function node
    - Output: payoff contributed by given function node, according to function table
  - Payoff for playing action a: weighted sum of function outputs from neighboring function nodes

RPS as BAGG-FNA

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0



Experiments: Random game generation ([11])

- Random additive polynomial sine BAGG-FNAs
  - Subgraph containing edges from action nodes to function nodes: Erdös-Rényi random graph
  - Subgraph containing edges from function nodes to action nodes: complete bipartite graph
  - Action weights generated according to normal distribution
  - Function nodes: sum of long-period sine function and low-degree polynomial
- Add noise to BAGG-FNA training data payoffs

Additional Hyperparameters

- Number of initial queries
- Number of initial training epochs
- Number of replicator dynamics mixtures
- Max regret (if intermediate regret check)
- ω_{mx}, the mixture resample factor
- Number of resample queries
- Number of retrain epochs
- Number of replicator dynamics iterations
- Number of resample/retrain iterations
- ω_{p}, the player count resample factor